Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Wednesday 11 November 2015 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

13 pages

8815 – 7201

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

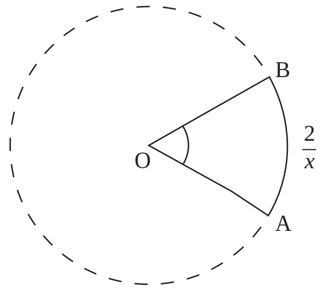
Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The following diagram shows a sector of a circle where $\text{A}\hat{\text{O}}\text{B} = x$ radians and the length of the arc $\text{AB} = \frac{2}{x}\text{cm}$.

Given that the area of the sector is $16\,\text{cm}^2$, find the length of the arc AB.



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2. [Maximum mark: 4]

Using integration by parts find $\int x \sin x \, dx$.

3. [Maximum mark: 6]

(a) Write down and simplify the expansion of $(2+x)^4$ in ascending powers of x. [3]

(b) Hence find the exact value of $(2.1)^4$. [3]

4. [Maximum mark: 6]

Consider the curve $y = \frac{1}{1-x}$, $x \in \mathbb{R}$, $x \neq 1$.

(a) Find $\frac{dy}{dx}$. [2]

(b) Determine the equation of the normal to the curve at the point $x = 3$ in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$. [4]

5. [Maximum mark: 4]

Use the substitution $u = \ln x$ to find the value of $\int_{e}^{e^2} \frac{dx}{x \ln x}$.

6. [Maximum mark: 7]

A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

(a) Darren plays first, find the probability that he wins. [4]

The game is now changed so that the ball chosen is replaced after each turn.
Darren still plays first.

(b) Show that the probability of Darren winning has not changed. [3]

7. [Maximum mark: 8]

A curve is defined by $xy = y^2 + 4$.

(a) Show that there is no point where the tangent to the curve is horizontal. [4]

(b) Find the coordinates of the points where the tangent to the curve is vertical. [4]

8. [Maximum mark: 8]

(a) Show that $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$. [1]

(b) Consider $f(x)=\sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x)=a^n\sin\left(ax+\frac{n\pi}{2}\right)$ where $n\in\mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

9. [Maximum mark: 7]

Solve the equation $\sin 2x-\cos 2x=1+\sin x-\cos x$ for $x\in[-\pi,\pi]$.

10. [Maximum mark: 6]

A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

Given that $a_{n-1} = -63$ and $a_n = 16$ find

(a) the degree of the polynomial; [4]

(b) the value of a_0. [2]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

(a) Solve the equation $z^3 = 8\mathrm{i}$, $z \in \mathbb{C}$ giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ **and** in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [6]

(b) Consider the complex numbers $z_1 = 1 + \mathrm{i}$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$.

(i) Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

(ii) Calculate z_1z_2 and write in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$.

(iii) Hence find the value of $\tan\frac{5\pi}{12}$ in the form $c + d\sqrt{3}$, where $c, d \in \mathbb{Z}$.

(iv) Find the smallest value $p > 0$ such that $(z_2)^p$ is a positive real number. [11]

12. [Maximum mark: 20]

Consider the function defined by $f(x) = x\sqrt{1 - x^2}$ on the domain $-1 \le x \le 1$.

(a) Show that f is an odd function. [2]

(b) Find $f'(x)$. [3]

(c) Hence find the x-coordinates of any local maximum or minimum points. [3]

(d) Find the range of f. [3]

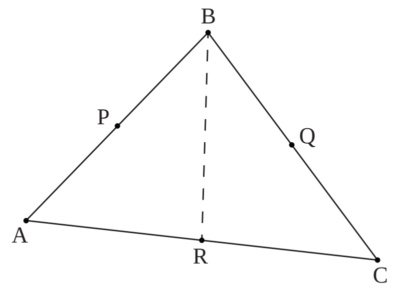
(e) Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

(f) Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \ge 0$. [4]

(g) Show that $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|\mathrm{d}x > \left|\int_{-1}^{1}x\sqrt{1-x^2}\,\mathrm{d}x\right|$. [2]

Do **not** write solutions on this page.

13. [Maximum mark: 23]



Consider the triangle ABC. The points P, Q and R are the midpoints of the line segments [AB], [BC] and [AC] respectively.

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

(a) Find $\overrightarrow{BR}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. [2]

(b) (i) Find a vector equation of the line that passes through B and R in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter λ.

(ii) Find a vector equation of the line that passes through A and Q in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and a parameter μ.

(iii) Hence show that $\overrightarrow{OG} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ given that G is the point where [BR] and [AQ] intersect. [9]

(c) Show that the line segment [CP] also includes the point G. [3]

The coordinates of the points A, B and C are $(1, 3, 1)$, $(3, 7, -5)$ and $(2, 2, 1)$ respectively.
A point X is such that [GX] is perpendicular to the plane ABC.

(d) Given that the tetrahedron ABCX has volume 12 units^3, find possible coordinates of X. [9]

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N15/5/MATHL/HP1/ENG/TZ0/XX/M

Markscheme

November 2015

Mathematics

Higher level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. arc length $= \frac{2}{x} = rx \left(\Rightarrow r = \frac{2}{x^2} \right)$ ***M1***

$16 = \frac{1}{2}\left(\frac{2}{x^2}\right)^2 x \left(\Rightarrow \frac{2}{x^3} = 16 \right)$ ***M1***

Note: Award M1s for attempts at the use of arc-length and sector-area formulae.

$x = \frac{1}{2}$ ***A1***

arc length $= 4(\text{cm})$ ***A1***

[4 marks]

2. attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x \, dx = x(-\cos x) + \int \cos x \, dx$ ***(A1)(A1)***

$= -x \cos x + \sin x + c$ ***A1***

Note: Only award final ***A1*** if $+c$ is seen.

[4 marks]

3. (a) $(2+x)^4 = 2^4 + 4 \cdot 2^3 x + 6 \cdot 2^2 x^2 + 4 \cdot 2x^3 + x^4$ ***M1(A1)***

Note: Award ***M1*** for an expansion, by whatever method, giving five terms in any order.

$= 16 + 32x + 24x^2 + 8x^3 + x^4$ ***A1***

Note: Award ***M1A1A0*** for correct expansion not given in ascending powers of x.

[3 marks]

(b) let $x = 0.1$ (in the binomial expansion) ***(M1)***

$2.1^4 = 16 + 3.2 + 0.24 + 0.008 + 0.0001$ ***(A1)***

$= 19.4481$ ***A1***

Note: At most one of the marks can be implied.

[3 marks]

Total [6 marks]

4. (a) $\frac{dy}{dx} = (1-x)^{-2} \left(= \frac{1}{(1-x)^2} \right)$ ***(M1)A1***

[2 marks]

continued…

Question 4 continued

(b) gradient of Tangent $= \frac{1}{4}$ ***(A1)***

gradient of Normal $= -4$ ***(M1)***

$y + \frac{1}{2} = -4(x-3)$ or attempt to find c in $y = mx + c$ ***M1***

$8x + 2y - 23 = 0$ ***A1***

[4 marks]

Total [6 marks]

5. **METHOD 1**

$\int_{e}^{e^2} \frac{dx}{x \ln x} = [\ln(\ln x)]_{e}^{e^2}$ ***(M1)A1***

$= \ln(\ln e^2) - \ln(\ln e) \; (= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

METHOD 2

$u = \ln x, \frac{du}{dx} = \frac{1}{x}$ ***M1***

$= \int_1^2 \frac{du}{u}$ ***A1***

$= [\ln u]_1^2$ or equivalent in x $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

6. (a) probability that Darren wins $P(W) + P(RRW) + P(RRRRW)$ ***(M1)***

Note: Only award ***M1*** if three terms are seen or are implied by the following numerical equivalent.

Note: Accept equivalent tree diagram for method mark.

$= \frac{2}{6} + \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} + \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \quad \left(= \frac{1}{3} + \frac{1}{5} + \frac{1}{15} \right)$ ***A2***

Note: ***A1*** for two correct.

$= \frac{3}{5}$ ***A1***

[4 marks]

continued…

Question 6 continued

(b) **METHOD 1**

the probability that Darren wins is given by
$P(W) + P(RRW) + P(RRRRW) + \ldots$ ***(M1)***

Note: Accept equivalent tree diagram with correctly indicated path for method mark.

$P(\text{Darren Win}) = \frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \ldots$

or $= \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \ldots\right)$ ***A1***

$= \frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)$ ***A1***

$= \frac{3}{5}$ ***AG***

[3 marks]

METHOD 2

$P(\text{Darren wins}) = P$

$P = \frac{1}{3} + \frac{4}{9}P$ ***M1A2***

$\frac{5}{9}P = \frac{1}{3}$

$P = \frac{3}{5}$ ***AG***

[3 marks]

Total [7 marks]

7. (a) $x\frac{dy}{dx} + y = 2y\frac{dy}{dx}$ ***M1A1***

a horizontal tangent occurs if $\frac{dy}{dx} = 0$ so $y = 0$ ***M1***

we can see from the equation of the curve that this solution is not possible $(0 = 4)$ and so there is not a horizontal tangent ***R1***

[4 marks]

continued…

Question 7 continued

(b) $\frac{dy}{dx} = \frac{y}{2y - x}$ or equivalent with $\frac{dx}{dy}$

the tangent is vertical when $2y = x$ ***M1***

substitute into the equation to give $2y^2 = y^2 + 4$ ***M1***

$y = \pm 2$ ***A1***

coordinates are $(4, 2), (-4, -2)$ ***A1***

[4 marks]

Total [8 marks]

8. (a) $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

(b) consider $n = 1$, $f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}\left(= a\left(a^k\cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior M and R marks have been awarded.

[7 marks]

Total [8 marks]

9. $(\sin 2x - \sin x) - (\cos 2x - \cos x) = 1$

attempt to use both double-angle formulae, in whatever form ***M1***

$(2\sin x \cos x - \sin x) - (2\cos^2 x - 1 - \cos x) = 1$

or $(2\sin x \cos x - \sin x) - \left(2\cos^2 x - \cos x\right) = 0$ for example ***A1***

> **Note:** Allow any rearrangement of the above equations.

$\sin x(2\cos x - 1) - \cos x(2\cos x - 1) = 0$

$(\sin x - \cos x)(2\cos x - 1) = 0$ ***(M1)***

$\tan x = 1$ and $\cos x = \frac{1}{2}$ ***A1A1***

> **Note:** These ***A*** marks are dependent on the ***M*** mark awarded for factorisation.

$x = -\frac{3\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$ ***A2***

> **Note:** Award ***A1*** for two correct answers, which could be for both tan or both cos solutions, for example.

[7 marks]

10. (a) the sum of the roots of the polynomial $= \frac{63}{16}$ ***(A1)***

$2\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{63}{16}$ ***M1A1***

> **Note:** The formula for the sum of a geometric sequence must be equated to a value for the ***M1*** to be awarded.

$1 - \left(\frac{1}{2}\right)^n = \frac{63}{64} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{1}{64}$

$n = 6$ ***A1***

[4 marks]

(b) $\frac{a_0}{a_n} = 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}, \ (a_n = 16)$ ***M1***

$a_0 = 16 \times 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}$

$a_0 = 2^{-5}\left(= \frac{1}{32}\right)$ ***A1***

[2 marks]

Total [6 marks]

Section B

11. (a) $z^3 = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(A1)***

attempt the use of De Moivre's Theorem in reverse ***M1***

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right);\ 2\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right);$

$2\left(\cos\left(\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(\frac{9\pi}{6}\right)\right)$ ***A2***

> **Note:** Accept cis form.

$z = \pm\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

> **Note:** Award ***A1*** for two correct solutions in each of the two lines above.

[6 marks]

(b) (i) $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$ ***A1A1***

(ii) $\left(z_2 = \left(\sqrt{3} + \mathrm{i}\right)\right)$

$z_1 z_2 = (1 + \mathrm{i})\left(\sqrt{3} + \mathrm{i}\right)$ ***M1***

$= \left(\sqrt{3} - 1\right) + \mathrm{i}\left(1 + \sqrt{3}\right)$ ***A1***

(iii) $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\right)$ ***M1A1***

$\tan\frac{5\pi}{12} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ ***A1***

$= 2 + \sqrt{3}$ ***M1A1***

> **Note:** Award final ***M1*** for an attempt to rationalise the fraction.

(iv) $z_2^{\,p} = 2^p\left(\mathrm{cis}\left(\frac{p\pi}{6}\right)\right)$ ***(M1)***

$z_2^{\,p}$ is a positive real number when $p = 12$ ***A1***

[11 marks]

Total [17 marks]

12. (a) $f(-x) = (-x)\sqrt{1-(-x)^2}$ *M1*

$= -x\sqrt{1-x^2}$

$= -f(x)$ *R1*

hence f is odd *AG*

[2 marks]

(b) $f'(x) = x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x + (1-x^2)^{\frac{1}{2}}$ *M1A1A1*

[3 marks]

(c) $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ *A1*

Note: This may be seen in part (b).

$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$ *M1*

$x = \pm\frac{1}{\sqrt{2}}$ *A1*

[3 marks]

(d) y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ *M1A1*

so range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ *A1*

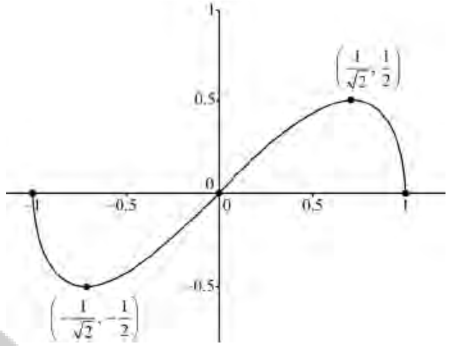
Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

continued…

Question 12 continued

(e)



Shape: The graph of an odd function, on the given domain, s-shaped, where the max(min) is the right(left) of 0.5(-0.5) *A1*

x-intercepts *A1*

turning points *A1*

[3 marks]

(f) area $= \int_0^1 x\sqrt{1-x^2}\,dx$ *(M1)*

attempt at "backwards chain rule" or substitution *M1*

$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\,dx$

$= \left[\frac{2}{3}(1-x^2)^{\frac{3}{2}} \cdot -\frac{1}{2}\right]_0^1$ *A1*

$= \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ *A1*

[4 marks]

(g) $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|dx > 0$ *R1*

$\left|\int_{-1}^{1} x\sqrt{1-x^2}\,dx\right| = 0$ *R1*

so $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|dx > \left|\int_{-1}^{1} x\sqrt{1-x^2}\,dx\right| = 0$ *AG*

[2 marks]

Total [20 marks]

13. (a) $\vec{BR} = \vec{BA} + \vec{AR} \ \left(= \vec{BA} + \frac{1}{2}\vec{AC}\right)$ ***(M1)***

$= (\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***A1***

[2 marks]

(b) (i) $r_{BR} = \boldsymbol{b} + \lambda\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= \frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c}\right)$ ***A1A1***

Note: Award ***A1A0*** if the r = is omitted in an otherwise correct expression/equation.

(ii) $\vec{AQ} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***(A1)***

$r_{AQ} = \boldsymbol{a} + \mu\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}\right)$ ***A1***

(iii) when $\vec{AQ}$ and $\vec{BP}$ intersect we will have $r_{BR} = r_{AQ}$ ***(M1)***

$\frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c} = (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}$

attempt to equate the coefficients of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ***M1***

(A1)

$$\left.\begin{array}{l}\frac{\lambda}{2} = 1 - \mu \\ 1 - \lambda = \frac{\mu}{2} \\ \frac{\lambda}{2} = \frac{\mu}{2}\end{array}\right\}$$

$\lambda = \frac{2}{3}$ or $\mu = \frac{2}{3}$ ***A1***

substituting parameters back into one of the equations ***M1***

$\vec{OG} = \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{a} + \left(1 - \frac{2}{3}\right)\boldsymbol{b} + \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{c} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

[9 marks]

continued…

Question 13 continued

(c) $\vec{CP} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$ ***(M1)A1***

so we have that $r_{CP} = \boldsymbol{c} + \beta\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}\right)$ and when $\beta = \frac{2}{3}$ the line passes through

the point G (*ie*, with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$) ***R1***

hence [AQ], [BR] and [CP] all intersect in G ***AG***

[3 marks]

continued…

Question 13 continued

(d) $\vec{OG} = \frac{1}{3}\left(\begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}3\\7\\-5\end{pmatrix} + \begin{pmatrix}2\\2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$ ***A1***

Note: This independent mark for the vector may be awarded wherever the vector is calculated.

$\vec{AB} \times \vec{AC} = \begin{pmatrix}2\\4\\-6\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$ ***M1A1***

$\vec{GX} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$ ***(M1)***

volume of Tetrahedron given by $\frac{1}{3} \times$ Area $\text{ABC} \times \text{GX}$

$= \frac{1}{3}\left(\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|\right) \times \text{GX} = 12$ ***(M1)(A1)***

Note: Accept alternative methods, for example the use of a scalar triple product.

$= \frac{1}{6}\sqrt{(-6)^2 + (-6)^2 + (-6)^2} \times \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = 12$ ***(A1)***

$= \frac{1}{6} 6\sqrt{3}\,|\alpha|\sqrt{3} = 12$

$\Rightarrow |\alpha| = 4$ ***A1***

Note: Condone absence of absolute value.

this gives us the position of X as $\begin{pmatrix}2\\4\\-1\end{pmatrix} \pm \begin{pmatrix}4\\4\\4\end{pmatrix}$

$\text{X}(6, 8, 3)$ or $(-2, 0, -5)$ ***A1***

Note: Award ***A1*** for either result.

[9 marks]

Total [23 marks]

N15/5/MATHL/HP3/ENG/TZ0/SE

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

3 pages

8815 – 7208

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f : x \to \begin{cases} 1 \quad , x < 0 \\ 1 - x, \; x \geq 0 \end{cases}$.

By considering limits, prove that f is

(a) continuous at $x = 0$; [2]

(b) not differentiable at $x = 0$. [3]

2. [Maximum mark: 10]

Let $f(x) = \mathrm{e}^x \sin x$.

(a) Show that $f''(x) = 2(f'(x) - f(x))$. [4]

(b) By further differentiation of the result in part (a), find the Maclaurin expansion of $f(x)$, as far as the term in x^5. [6]

3. [Maximum mark: 11]

(a) Prove by induction that $n! > 3^n$, for $n \geq 7$, $n \in \mathbb{Z}$. [5]

(b) Hence use the comparison test to prove that the series $\sum_{r=1}^{\infty} \frac{2^r}{r!}$ converges. [6]

4. [Maximum mark: 14]

Consider the function $f(x) = \frac{1}{1 + x^2}$, $x \in \mathbb{R}$.

(a) Illustrate graphically the inequality, $\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\,\mathrm{d}x < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$. [3]

(b) Use the inequality in part (a) to find a lower and upper bound for π. [5]

(c) Show that $\sum_{r=0}^{n-1} (-1)^r x^{2r} = \frac{1 + (-1)^{n-1} x^{2n}}{1 + x^2}$. [2]

(d) Hence show that $\pi = 4\left(\sum_{r=0}^{n-1} \frac{(-1)^r}{2r + 1} - (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1 + x^2}\,\mathrm{d}x\right)$. [4]

5. [Maximum mark: 20]

The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1, 0)$ and are defined by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

(a) Show that the tangent to the curve $y = f(x)$ at the point $(1, 0)$ is normal to the curve $y = g(x)$ at the point $(1, 0)$. [2]

(b) Find $g(x)$. [6]

(c) Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

(d) Explain why $y = f(x)$ cannot cross the isocline $x - y^2 = 0$, for $x > 1$. [3]

(e) (i) Sketch the isoclines $x - y^2 = -2, 0, 1$.

(ii) On the same set of axes, sketch the graph of f. [4]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – discrete mathematics

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

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- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

5 pages

8815 – 7207

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 11]

(a) The distances by road, in kilometres, between towns in Switzerland are shown in the following table.

	Basel	**Berne**	**Geneva**	**Lugano**	**Sion**	**Zurich**
Basel		100	260	260	250	85
Berne	100		170	275	155	125
Geneva	260	170		440	160	290
Lugano	260	275	440		255	210
Sion	250	155	160	255		275
Zurich	85	125	290	210	275	

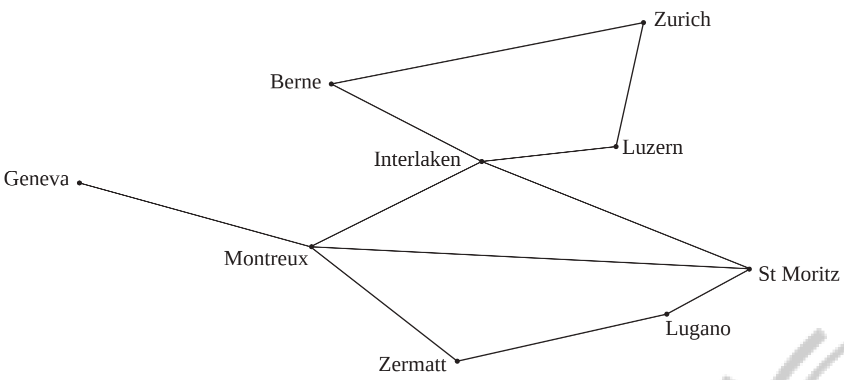
A cable television company wishes to connect the six towns placing cables along the road system.

Use Kruskal’s algorithm to find the minimum length of cable needed to connect the six towns. [5]

(This question continues on the following page)

(Question 1 continued)

(b) Visitors to Switzerland can visit some principal locations for tourism by using a network of scenic railways as represented by the following graph:



(i) State whether the graph has any Hamiltonian paths or Hamiltonian cycles, justifying your answers.

(ii) State whether the graph has any Eulerian trails or Eulerian circuits, justifying your answers.

(iii) The tourist board would like to make it possible to arrive in Geneva, travel all the available scenic railways, exactly once, and depart from Zurich. Find which locations would need to be connected by a further scenic railway in order to make this possible. [6]

2. [Maximum mark: 13]

A recurrence relation is given by $u_{n+1} + 2u_n + 1 = 0$, $u_1 = 4$.

(a) Use the recurrence relation to find u_2. [1]

(b) Find an expression for u_n in terms of n. [6]

A second recurrence relation, where $v_1 = u_1$ and $v_2 = u_2$, is given by $v_{n+1} + 2v_n + v_{n-1} = 0$, $n \geq 2$.

(c) Find an expression for v_n in terms of n. [6]

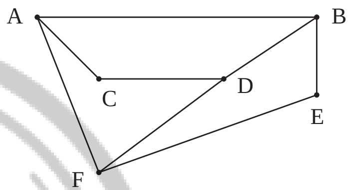
3. [Maximum mark: 13]

(a) Show that there are exactly two solutions to the equation $1982 = 36a + 74b$, with $a, b \in \mathbb{N}$. [8]

(b) Hence, or otherwise, find the remainder when 1982^{1982} is divided by 37. [5]

4. [Maximum mark: 13]

The following diagram shows the graph G.



(a) Show that G is bipartite. [2]

(b) State which two vertices should be joined to make G equivalent to $K_{3,3}$. [1]

In a planar graph the degree of a face is defined as the number of edges adjacent to that face.

(c) (i) Write down the degree of each of the four faces of G.

(ii) Explain why the sum of the degrees of all the faces is twice the number of edges. [2]

H is a simple connected planar bipartite graph with e edges, f faces, v vertices and $v \geq 3$.

(d) Explain why there can be no face in H of degree

(i) one;

(ii) two;

(iii) three. [3]

(This question continues on the following page)

(Question 4 continued)

(e) Hence prove that for H

(i) $e \geq 2f$;

(ii) $e \leq 2v - 4$. [3]

(f) Hence prove that $K_{3,3}$ is not planar. [2]

5. [Maximum mark: 10]

(a) Given a sequence of non negative integers $\{a_r\}$ show that

(i) $\sum_{r=0}^{n} a_r (x+1)^r (\text{mod } x) = \sum_{r=0}^{n} a_r (\text{mod } x)$ where $x \in \{2, 3, 4...\}$.

(ii) $\sum_{r=0}^{n} (3a_{2r+1} + a_{2r}) 9^r = \sum_{r=0}^{2n+1} a_r 3^r$. [5]

(b) Hence determine whether the base 3 number 22010112200201 is divisible by 8 . [5]

Mathematics
Higher level
Paper 3 – sets, relations and groups

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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1. [Maximum mark: 5]

Given the sets A and B, use the properties of sets to prove that $A \cup (B' \cup A)' = A \cup B$, justifying each step of the proof.

2. [Maximum mark: 14]

The function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f : x \to \begin{cases} 1 \ , & x \geq 0 \\ -1 \ , & x < 0 \end{cases}$.

(a) Prove that f is

(i) not injective;

(ii) not surjective. [4]

The relation R is defined for $a, b \in \mathbb{R}$ so that aRb if and only if $f(a) \times f(b) = 1$.

(b) Show that R is an equivalence relation. [8]

(c) State the equivalence classes of R. [2]

3. [Maximum mark: 10]

The set of all permutations of the elements $1, 2, \ldots 10$ is denoted by H and the binary operation $\circ$ represents the composition of permutations.
The permutation $p = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$ generates the subgroup $\{G, \circ\}$ of the group $\{H, \circ\}$.

(a) Find the order of $\{G, \circ\}$. [2]

(b) State the identity element in $\{G, \circ\}$. [1]

(c) Find

(i) $p \circ p$;

(ii) the inverse of $p \circ p$. [4]

(d) (i) Find the maximum possible order of an element in $\{H, \circ\}$.

(ii) Give an example of an element with this order. [3]

4. [Maximum mark: 18]

The binary operation $*$ is defined on the set $T = \{0, 2, 3, 4, 5, 6\}$ by $a * b = (a + b - ab)(\text{mod}\, 7)$, $a, b \in T$.

(a) Copy and complete the following Cayley table for $\{T, *\}$. [4]

$*$	**0**	**2**	**3**	**4**	**5**	**6**
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3	6				
4	4	5				
5	5	4				
6	6	3				

(b) Prove that $\{T, *\}$ forms an Abelian group. [7]

(c) Find the order of each element in T. [4]

(d) Given that $\{H, *\}$ is the subgroup of $\{T, *\}$ of order 2, partition T into the left cosets with respect to H. [3]

5. [Maximum mark: 13]

A group $\{D, \times_3\}$ is defined so that $D = \{1, 2\}$ and $\times_3$ is multiplication modulo 3.

A function $f: \mathbb{Z} \to D$ is defined as $f: x \mapsto \begin{cases} 1, & x \text{ is even} \\ 2, & x \text{ is odd} \end{cases}$.

(a) Prove that the function f is a homomorphism from the group $\{\mathbb{Z}, +\}$ to $\{D, \times_3\}$. [6]

(b) Find the kernel of f. [3]

(c) Prove that $\{\text{Ker}(f), +\}$ is a subgroup of $\{\mathbb{Z}, +\}$. [4]

Mathematics
Higher level
Paper 3 – statistics and probability

Wednesday 18 November 2015 (afternoon)

1 hour

Instructions to candidates

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- Answer all the questions.
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- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

8815 – 7210

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1. [Maximum mark: 7]

It is known that the standard deviation of the heights of men in a certain country is 15.0 cm.

(a) One hundred men from that country, selected at random, had their heights measured. The mean of this sample was 185 cm. Calculate a 95% confidence interval for the mean height of the population. [3]

(b) A second random sample of size n is taken from the same population. Find the minimum value of n needed for the width of a 95% confidence interval to be less than 3 cm. [4]

2. [Maximum mark: 11]

The strength of beams compared against the moisture content of the beam is indicated in the following table. You should assume that strength and moisture content are each normally distributed.

Strength	21.1	22.7	23.1	21.5	22.4	22.6	21.1	21.7	21.0	21.4
Moisture content	11.1	8.9	8.8	8.9	8.8	9.9	10.7	10.5	10.5	10.7

(a) Determine the product moment correlation coefficient for these data. [2]

(b) Perform a two-tailed test, at the 5% level of significance, of the hypothesis that strength is independent of moisture content. [5]

(c) If the moisture content of a beam is found to be 9.5, use the appropriate regression line to estimate the strength of the beam. [4]

3. [Maximum mark: 9]

Two students are selected at random from a large school with equal numbers of boys and girls. The boys' heights are normally distributed with mean 178 cm and standard deviation 5.2 cm, and the girls' heights are normally distributed with mean 169 cm and standard deviation 5.4 cm.

Calculate the probability that the taller of the two students selected is a boy.

4. [Maximum mark: 22]

A discrete random variable U follows a geometric distribution with $p = \frac{1}{4}$.

(a) Find $F(u)$, the cumulative distribution function of U, for $u = 1, 2, 3 \ldots$ [3]

(b) Hence, or otherwise, find the value of $P(U > 20)$. [2]

(c) Prove that the probability generating function of U is given by $G_u(t) = \frac{t}{4 - 3t}$. [4]

(d) Given that $U_i \sim \text{Geo}\left(\frac{1}{4}\right)$, $i = 1, 2, 3$, and that $V = U_1 + U_2 + U_3$, find

(i) $\text{E}(V)$;

(ii) $\text{Var}(V)$;

(iii) $G_v(t)$, the probability generating function of V. [6]

A third random variable W, has probability generating function $G_w(t) = \frac{1}{(4 - 3t)^3}$.

(e) By differentiating $G_w(t)$, find $\text{E}(W)$. [4]

(f) Prove that $V = W + 3$. [3]

5. [Maximum mark: 11]

A biased cubical die has its faces labelled 1, 2, 3, 4, 5 and 6. The probability of rolling a 6 is p, with equal probabilities for the other scores.

The die is rolled once, and the score X_1 is noted.

(a) (i) Find $\mathrm{E}(X_1)$.

(ii) Hence obtain an unbiased estimator for p. [4]

The die is rolled a second time, and the score X_2 is noted.

(b) (i) Show that $k\left(X_1 - 3\right) + \left(\frac{1}{3} - k\right)\left(X_2 - 3\right)$ is also an unbiased estimator for p for all values of $k \in \mathbb{R}$.

(ii) Find the value for k, which maximizes the efficiency of this estimator. [7]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Thursday 10 November 2016 (afternoon)

2 hours

Candidate session number

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

13 pages

8816 – 7201

16EP01