**5.** [Maximum mark: 11]

A biased cubical die has its faces labelled $1$, $2$, $3$, $4$, $5$ and $6$. The probability of rolling a $6$ is $p$, with equal probabilities for the other scores.

The die is rolled once, and the score $X_1$ is noted.

(a) (i) Find $\mathrm{E}(X_1)$.

(ii) Hence obtain an unbiased estimator for $p$. [4]

The die is rolled a second time, and the score $X_2$ is noted.

(b) (i) Show that $k(X_1 - 3) + \left(\frac{1}{3} - k\right)(X_2 - 3)$ is also an unbiased estimator for $p$ for all values of $k \in \mathbb{R}$.

(ii) Find the value for $k$, which maximizes the efficiency of this estimator. [7]





**Mathematics**
**Higher level**
**Paper 1**

Thursday 10 November 2016 (afternoon)

2 hours

Candidate session number

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

**Instructions to candidates**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks]**.

13 pages

8816 – 7201


16EP01

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Find the coordinates of the point of intersection of the planes defined by the equations $x+y+z=3$, $x-y+z=5$ and $x+y+2z=6$.





**2.** [Maximum mark: 4]

The faces of a fair six-sided die are numbered $1, 2, 2, 4, 4, 6$. Let $X$ be the discrete random variable that models the score obtained when this die is rolled.

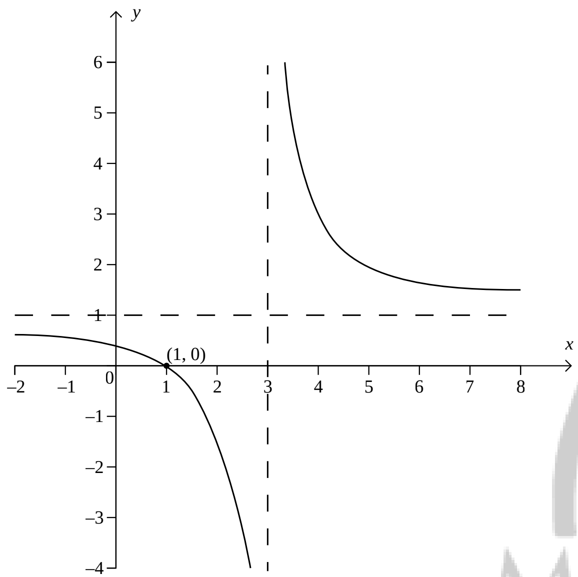
(a) Complete the probability distribution table for $X$. [2]

| $x$ | | | | |
|---|---|---|---|---|
| $\mathrm{P}(X=x)$ | | | | |

(b) Find the expected value of $X$. [2]

**3.** [Maximum mark: 4]

A rational function is defined by $f(x) = a + \dfrac{b}{x - c}$ where the parameters $a, b, c \in \mathbb{Z}$ and $x \in \mathbb{R} \setminus \{c\}$. The following diagram represents the graph of $y = f(x)$.




Using the information on the graph,

(a) state the value of $a$ and the value of $c$; [2]

(b) find the value of $b$. [2]





**4.** [Maximum mark: 5]

Consider the vectors $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{j} + 2\boldsymbol{k}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$. [2]

(b) Hence find the Cartesian equation of the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and passing through the point $(1, 0, -1)$. [3]

**5.** [Maximum mark: 6]

The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots $\alpha$ and $\beta$ such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number $k$.





**6.** [Maximum mark: 7]

The sum of the first $n$ terms of a sequence $\{u_n\}$ is given by $S_n = 3n^2 - 2n$, where $n \in \mathbb{Z}^+$.

(a) Write down the value of $u_1$. [1]

(b) Find the value of $u_6$. [2]

(c) Prove that $\{u_n\}$ is an arithmetic sequence, stating clearly its common difference. [4]

**7.** [Maximum mark: 5]

Solve the equation $4^x + 2^{x+2} = 3$.





**8.** [Maximum mark: 6]

Consider the lines $l_1$ and $l_2$ defined by

$$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \text{ and } l_2 : \frac{6-x}{3} = \frac{y-2}{4} = 1 - z \text{ where } a \text{ is a constant.}$$

Given that the lines $l_1$ and $l_2$ intersect at a point P,

(a) find the value of $a$; [4]

(b) determine the coordinates of the point of intersection P. [2]

**9.** [Maximum mark: 9]

A curve has equation $3x - 2y^2 e^{x-1} = 2$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$. [5]

(b) Find the equations of the tangents to this curve at the points where the curve intersects the line $x = 1$. [4]





**10.** [Maximum mark: 9]

Consider two events $A$ and $B$ defined in the same sample space.

(a) Show that $P(A \cup B) = P(A) + P(A' \cap B)$. [3]

(b) Given that $P(A \cup B) = \frac{4}{9}$, $P(B \mid A) = \frac{1}{3}$ and $P(B \mid A') = \frac{1}{6}$,

(i) show that $P(A) = \frac{1}{3}$;

(ii) hence find $P(B)$. [6]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**11.** [Maximum mark: 22]

Let $y = e^x \sin x$.

(a) Find an expression for $\frac{dy}{dx}$. [2]

(b) Show that $\frac{d^2y}{dx^2} = 2e^x \cos x$. [2]

Consider the function $f$ defined by $f(x) = e^x \sin x$, $0 \le x \le \pi$.

(c) Show that the function $f$ has a local maximum value when $x = \frac{3\pi}{4}$. [2]

(d) Find the $x$-coordinate of the point of inflexion of the graph of $f$. [2]

(e) Sketch the graph of $f$, clearly indicating the position of the local maximum point, the point of inflexion and the axes intercepts. [3]

(f) Find the area of the region enclosed by the graph of $f$ and the $x$-axis. [6]

The curvature at any point $(x, y)$ on a graph is defined as $\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$.

(g) Find the value of the curvature of the graph of $f$ at the local maximum point. [3]

(h) Find the value $\kappa$ for $x = \frac{\pi}{2}$ and comment on its meaning with respect to the shape of the graph. [2]





Do **not** write solutions on this page.

**12.** [Maximum mark: 19]

Let $\omega$ be one of the non-real solutions of the equation $z^3 = 1$.

(a) Determine the value of

(i) $1 + \omega + \omega^2$;

(ii) $1 + \omega^* + (\omega^*)^2$. [4]

(b) Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$. [4]

Consider the complex numbers $p = 1 - 3i$ and $q = x + (2x + 1)i$, where $x \in \mathbb{R}$.

(c) Find the values of $x$ that satisfy the equation $|p| = |q|$. [5]

(d) Solve the inequality $\text{Re}(pq) + 8 < (\text{Im}(pq))^2$. [6]

**13.** [Maximum mark: 19]

(a) Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}$. [2]

(b) Show that $\frac{1 - \cos 2x}{2\sin x} \equiv \sin x$, $x \ne k\pi$ where $k \in \mathbb{Z}$. [2]

(c) Use the principle of mathematical induction to prove that
$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1 - \cos 2nx}{2\sin x}$, $n \in \mathbb{Z}^+$, $x \ne k\pi$ where $k \in \mathbb{Z}$. [9]

(d) Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. [6]

Please **do not** write on this page.

Answers written on this page will not be marked.

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Diploma Programme
Programme du diplôme
Programa del Diploma

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Answers written on this page will not be marked.

# Markscheme

## November 2016

## Mathematics

## Higher level

## Paper 1

19 pages





International Baccalaureate
Baccalauréat International
Bachillerato Internacional

**Instructions to Examiners**

**Abbreviations**

***M*** Marks awarded for attempting to use a valid **Method**; working must be seen.

***(M)*** Marks awarded for **Method**; may be implied by **correct** subsequent working.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***(A)*** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

***R*** Marks awarded for clear **Reasoning**.

***N*** Marks awarded for **correct** answers if **no** working shown.

***AG*** Answer given in the question and so no marks are awarded.

**Using the markscheme**

**1 General**

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

| | **Correct answer seen** | **Further working seen** | **Action** |
|---|---|---|---|
| **1.** | $8\sqrt{2}$ | $5.65685...$ *(incorrect decimal value)* | Award the final ***A1*** *(ignore the further working)* |
| **2.** | $\frac{1}{4}\sin 4x$ | $\sin x$ | Do not award the final ***A1*** |
| **3.** | $\log a - \log b$ | $\log(a-b)$ | Do not award the final ***A1*** |

**3 *N* marks**

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

**4 Implied marks**

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

**5 Follow through marks**

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.



**6 Misread**

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

**7 Discretionary marks *(d)***

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

**8 Alternative methods**

*Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.*

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

**9 Alternative forms**

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

**Example**: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

**10 Accuracy of Answers**

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

**11 Crossed out work**

*If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.*

**12 Calculators**

*No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.*

**13 More than one solution**

*Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.*

**14. Candidate work**

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.



## Section A

**1. METHOD 1**

for eliminating one variable from two equations ***(M1)***

*eg*, $\begin{cases}(x+y+z=3)\\ 2x+2z=8\\ 2x+3z=11\end{cases}$ ***A1A1***

for finding correctly one coordinate

*eg*, $\Rightarrow \begin{cases}(x+y+z=3)\\ (2x+2z=8)\\ z=3\end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases}x=1\\ y=-1\\ z=3\end{cases}$

the intersection point has coordinates $(1, -1, 3)$

**METHOD 2**

for eliminating two variables from two equations or using row reduction ***(M1)***

*eg*, $\begin{cases}(x+y+z=3)\\ -2y=2\\ z=3\end{cases}$ **or** $\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&2\\0&0&1&3\end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases}x=1\\ y=-1\\ (z=3)\end{cases}$ **or** $\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&3\end{array}\right)$

the intersection point has coordinates $(1, -1, 3)$

*continued…*

*Question 1 continued*

**METHOD 3**

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$$ ***(A1)***

attempt to use Cramer's rule ***M1***

$$x = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 5 & -1 & 1 \\ 6 & 1 & 2 \end{vmatrix}}{-2} = \frac{-2}{-2} = 1$$ ***A1***

$$y = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 2 \end{vmatrix}}{-2} = \frac{2}{-2} = -1$$ ***A1***

$$z = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$$ ***A1***

**Note:** Award ***M1*** only if candidate attempts to determine at least one of the variables using this method.

***[5 marks]***

**2.** (a)

| $x$ | 1 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\text{P}(X = x)$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

***A1A1***

**Note:** Award ***A1*** for each correct row.

***[2 marks]***

(b) $\text{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{6}$ ***(M1)***

$= \frac{19}{6} \left( = 3\frac{1}{6} \right)$ ***A1***

**Note:** If the probabilities in (a) are not values between 0 and 1 or lead to $\text{E}(X) > 6$ award ***M1A0*** to correct method using the incorrect probabilities; otherwise allow ***FT*** marks.

***[2 marks]***

***Total [4 marks]***



**3.** (a) $a = 1$ ***A1***

$c = 3$ ***A1***

***[2 marks]***

(b) use the coordinates of $(1, 0)$ on the graph ***M1***

$f(1) = 0 \Rightarrow 1 + \frac{b}{1-3} = 0 \Rightarrow b = 2$ ***A1***

***[2 marks]***

***Total [4 marks]***

**4.** (a) $\boldsymbol{a} \times \boldsymbol{b} = -12\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ ***(M1)A1***

***[2 marks]***

(b) **METHOD 1**

$-12x - 2y - 3z = d$ ***M1***

$-12 \times 1 - 2 \times 0 - 3(-1) = d$ ***(M1)***

$\Rightarrow d = -9$ ***A1***

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

**METHOD 2**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix}$$ ***M1A1***

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$) ***A1***

***[3 marks]***

***Total [5 marks]***

**5.** $\alpha + \beta = 2k$ ***A1***

$\alpha\beta = k - 1$ ***A1***

$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$

$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ ***A1***

attempt to solve quadratic ***(M1)***

$k = 1, -\frac{1}{2}$ ***A1***

***[6 marks]***

**6.** (a) $u_1 = 1$ ***A1***

***[1 mark]***

(b) $u_6 = S_6 - S_5 = 31$ ***M1A1***

***[2 marks]***

(c) $u_n = S_n - S_{n-1}$ ***M1***

$= \left(3n^2 - 2n\right) - \left(3(n-1)^2 - 2(n-1)\right)$

$= \left(3n^2 - 2n\right) - \left(3n^2 - 6n + 3 - 2n + 2\right)$

$= 6n - 5$ ***A1***

$d = u_{n+1} - u_n$ ***R1***

$= 6n + 6 - 5 - 6n + 5$

$= \left(6(n+1) - 5\right) - (6n - 5)$

$= 6$ (constant) ***A1***

**Notes:** Award ***R1*** only if candidate provides a clear argument that proves that the difference between **ANY** two consecutive terms of the sequence is constant. Do not accept examples involving particular terms of the sequence nor circular reasoning arguments (*eg* use of formulas of APs to prove that it is an AP). Last ***A1*** is independent of ***R1***.

***[4 marks]***

***Total [7 marks]***

**7.** attempt to form a quadratic in $2^x$ ***M1***

$\left(2^x\right)^2 + 4 \cdot 2^x - 3 = 0$ ***A1***

$2^x = \frac{-4 \pm \sqrt{16 + 12}}{2} \left(= -2 \pm \sqrt{7}\right)$ ***M1***

$2^x = -2 + \sqrt{7}$ (as $-2 - \sqrt{7} < 0$) ***R1***

$x = \log_2\left(-2 + \sqrt{7}\right)$ $\left(x = \frac{\ln\left(-2 + \sqrt{7}\right)}{\ln 2}\right)$ ***A1***

**Note:** Award ***R0 A1*** if final answer is $x = \log_2\left(-2 \pm \sqrt{7}\right)$.

***[5 marks]***



**8.** (a) **METHOD 1**

$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3 + \beta \\ y = -2 + 4\beta \\ z = a + 2\beta \end{cases}$ ***M1***

$\frac{6 - (-3 + \beta)}{3} = \frac{(-2 + 4\beta) - 2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$ ***M1A1***

$\frac{6 - (-3 + \beta)}{3} = 1 - (a + 2\beta) \Rightarrow 2 = -5 - a \Rightarrow a = -7$ ***A1***

**METHOD 2**

$\begin{cases} -3 + \beta = 6 - 3\lambda \\ -2 + 4\beta = 4\lambda + 2 \\ a + 2\beta = 1 - \lambda \end{cases}$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1 - \lambda - 2\beta = -7$ ***A1***

***[4 marks]***

(b) $\overrightarrow{\text{OP}} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$ ***A1***

$\therefore \text{P}(0, 10, -1)$

***[2 marks]***

***Total [6 marks]***

**9.** (a) attempt to differentiate implicitly ***M1***

$3 - \left(4y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\right)\mathrm{e}^{x-1} = 0$ ***A1A1A1***

**Note:** Award ***A1*** for correctly differentiating each term.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3 \cdot \mathrm{e}^{1-x} - 2y^2}{4y}$ ***A1***

**Note:** This final answer may be expressed in a number of different ways.

***[5 marks]***



*continued…*

*Question 9 continued*

(b) $3-2y^2=2 \Rightarrow y^2=\frac{1}{2} \Rightarrow y=\pm\sqrt{\frac{1}{2}}$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3-2\cdot\frac{1}{2}}{\pm 4\sqrt{\frac{1}{2}}}=\pm\frac{\sqrt{2}}{2}$ ***M1***

at $\left(1, \sqrt{\frac{1}{2}}\right)$ the tangent is $y-\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}(x-1)$ and ***A1***

at $\left(1, -\sqrt{\frac{1}{2}}\right)$ the tangent is $y+\sqrt{\frac{1}{2}}=-\frac{\sqrt{2}}{2}(x-1)$ ***A1***

**Note:** These equations simplify to $y=\pm\frac{\sqrt{2}}{2}x$.

**Note:** Award ***A0M1A1A0*** if just the positive value of $y$ is considered and just one tangent is found.

***[4 marks]***

***Total [9 marks]***

**10** (a) **METHOD 1**

$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ ***M1***

$=\mathrm{P}(A)+\mathrm{P}(A\cap B)+\mathrm{P}(A'\cap B)-\mathrm{P}(A\cap B)$ ***M1A1***

$=\mathrm{P}(A)+\mathrm{P}(A'\cap B)$ ***AG***

**METHOD 2**

$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ ***M1***

$=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A|B)\times\mathrm{P}(B)$ ***M1***

$=\mathrm{P}(A)+\left(1-\mathrm{P}(A|B)\right)\times\mathrm{P}(B)$

$=\mathrm{P}(A)+\mathrm{P}(A'|B)\times\mathrm{P}(B)$ ***A1***

$=\mathrm{P}(A)+\mathrm{P}(A'\cap B)$ ***AG***

***[3 marks]***

(b) (i) use $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(A'\cap B)$ and $\mathrm{P}(A'\cap B)=\mathrm{P}(B\mid A')\mathrm{P}(A')$ ***(M1)***

$\frac{4}{9}=\mathrm{P}(A)+\frac{1}{6}\left(1-\mathrm{P}(A)\right)$ ***A1***

$8=18\mathrm{P}(A)+3\left(1-\mathrm{P}(A)\right)$ ***M1***

$\mathrm{P}(A)=\frac{1}{3}$ ***AG***

(ii) **METHOD 1**

$\mathrm{P}(B)=\mathrm{P}(A\cap B)+\mathrm{P}(A'\cap B)$ ***M1***

$=\mathrm{P}(B\mid A)\mathrm{P}(A)+\mathrm{P}(B\mid A')\mathrm{P}(A')$ ***M1***

$=\frac{1}{3}\times\frac{1}{3}+\frac{1}{6}\times\frac{2}{3}=\frac{2}{9}$ ***A1***

**METHOD 2**

$\mathrm{P}(A\cap B)=\mathrm{P}(B\mid A)\mathrm{P}(A)\Rightarrow\mathrm{P}(A\cap B)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$ ***M1***

$\mathrm{P}(B)=\mathrm{P}(A\cup B)+\mathrm{P}(A\cap B)-\mathrm{P}(A)$ ***M1***

$\mathrm{P}(B)=\frac{4}{9}+\frac{1}{9}-\frac{1}{3}=\frac{2}{9}$ ***A1***

***[6 marks]***

***Total [9 marks]***

## Section B

**11.** (a) $\frac{dy}{dx} = e^x \sin x + e^x \cos x \left(= e^x (\sin x + \cos x)\right)$ ***M1A1***

***[2 marks]***

(b) $\frac{d^2y}{dx^2} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$ ***M1A1***

$= 2e^x \cos x$ ***AG***

***[2 marks]***

(c) $\frac{dy}{dx} = e^{\frac{3\pi}{4}}\left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) = 0$ ***R1***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4} < 0$ ***R1***

hence maximum at $x = \frac{3\pi}{4}$ ***AG***

***[2 marks]***

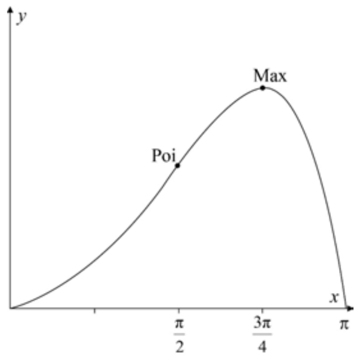
(d) $\frac{d^2y}{dx^2} = 0 \Rightarrow 2e^x \cos x = 0$ ***M1***

$\Rightarrow x = \frac{\pi}{2}$ ***A1***

| **Note:** Award ***M1A0*** if extra zeros are seen. |
|---|

***[2 marks]***

(e)




correct shape and correct domain ***A1***

max at $x = \frac{3\pi}{4}$, point of inflexion at $x = \frac{\pi}{2}$ ***A1***

zeros at $x = 0$ and $x = \pi$ ***A1***

| **Note:** Penalize incorrect domain with first ***A*** mark; allow ***FT*** from (d) on extra points of inflexion. |
|---|

***[3 marks]***

*continued…*



*Question 11 continued*

(f) **EITHER**

$\int_0^\pi e^x \sin x\, dx = \left[e^x \sin x\right]_0^\pi - \int_0^\pi e^x \cos x\, dx$ ***M1A1***

$\int_0^\pi e^x \sin x\, dx = \left[e^x \sin x\right]_0^\pi - \left(\left[e^x \cos x\right]_0^\pi + \int_0^\pi e^x \sin x\, dx\right)$ ***A1***

**OR**

$\int_0^\pi e^x \sin x\, dx = \left[-e^x \cos x\right]_0^\pi + \int_0^\pi e^x \cos x\, dx$ ***M1A1***

$\int_0^\pi e^x \sin x\, dx = \left[-e^x \cos x\right]_0^\pi + \left(\left[e^x \sin x\right]_0^\pi - \int_0^\pi e^x \sin x\, dx\right)$ ***A1***

**THEN**

$\int_0^\pi e^x \sin x\, dx = \frac{1}{2}\left(\left[e^x \sin x\right]_0^\pi - \left[e^x \cos x\right]_0^\pi\right)$ ***M1A1***

$\int_0^\pi e^x \sin x\, dx = \frac{1}{2}\left(e^\pi + 1\right)$ ***A1***

***[6 marks]***

(g) $\frac{dy}{dx} = 0$ ***(A1)***

$\frac{d^2y}{dx^2} = 2e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4} = -\sqrt{2}e^{\frac{3\pi}{4}}$ ***(A1)***

$\kappa = \frac{\left|-\sqrt{2}e^{\frac{3\pi}{4}}\right|}{1} = \sqrt{2}e^{\frac{3\pi}{4}}$ ***A1***

***[3 marks]***

(h) $\kappa = 0$ ***A1***

the graph is approximated by a straight line ***R1***

***[2 marks]***

***Total [22 marks]***

**12.** (a) (i) **METHOD 1**

$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega \neq 1$ ***R1***

**METHOD 2**

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\frac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+\left(\omega^*\right)^2=0$ ***A2***

***[4 marks]***

(b) $\left(\omega-3\omega^2\right)\left(\omega^2-3\omega\right)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1***

**EITHER**

$=-3\omega^2\left(\omega^2+\omega+1\right)+13\omega^3$ ***M1***

$=-3\omega^2\times 0+13\times 1$ ***A1***

**OR**

$=-3\omega+10-3\omega^2=-3\left(\omega^2+\omega+1\right)+13$ ***M1***

$=-3\times 0+13$ ***A1***

**OR**

substitution by $\omega=\frac{-1\pm\sqrt{3}\,\mathrm{i}}{2}$ in any form ***M1***

numerical values of each term seen ***A1***

**THEN**

$=13$ ***AG***

***[4 marks]***

(c) $|p|=|q| \Rightarrow \sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)***

$5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)***

$x=1,\ x=-\frac{9}{5}$ ***A1***

***[5 marks]***

*continued…*



*Question 12 continued*

(d) $pq=(1-3\mathrm{i})\left(x+(2x+1)\mathrm{i}\right)=(7x+3)+(1-x)\mathrm{i}$ ***M1A1***

$\mathrm{Re}(pq)+8<\left(\mathrm{Im}(pq)\right)^2 \Rightarrow (7x+3)+8<(1-x)^2$ ***M1***

$\Rightarrow x^2-9x-10>0$ ***A1***

$\Rightarrow (x+1)(x-10)>0$ ***M1***

$x<-1,\ x>10$ ***A1***

***[6 marks]***

***Total [19 marks]***

**13.** (a) $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$ ***(M1)A1***

**Note:** Award ***M1*** for 5 equal terms with + or – signs.

***[2 marks]***

(b) $\frac{1-\cos 2x}{2\sin x}\equiv\frac{1-\left(1-2\sin^2 x\right)}{2\sin x}$ ***M1***

$\equiv\frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv \sin x$ ***AG***

***[2 marks]***

*continued…*

*Question 13 continued*

(c) let $\mathrm{P}(n): \sin x + \sin 3x + \ldots + \sin(2n-1)x \equiv \dfrac{1-\cos 2nx}{2\sin x}$

if $n = 1$

$\mathrm{P}(1)$: $\dfrac{1-\cos 2x}{2\sin x} \equiv \sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x + \sin 3x + \ldots + \sin(2k-1)x \equiv \dfrac{1-\cos 2kx}{2\sin x}$ ***M1***

| |
|---|
| **Notes:** Only award ***M1*** if the words "assume" and "true" appear. Do not award ***M1*** for "let $n = k$" only. Subsequent marks are independent of this ***M1***. |

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1): \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x \equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

$LHS = \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x$ ***M1***

$\equiv \dfrac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x$ ***A1***

$\equiv \dfrac{1-\cos 2kx + 2\sin x\sin(2k+1)x}{2\sin x}$

$\equiv \dfrac{1-\cos 2kx + 2\sin x\cos x\sin 2kx + 2\sin^2 x\cos 2kx}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\left(1-2\sin^2 x\right)\cos 2kx - \sin 2x\sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv \dfrac{1-\left(\cos 2x\cos 2kx - \sin 2x\sin 2kx\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos\left(2kx+2x\right)}{2\sin x}$ ***A1***

$\equiv \dfrac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

| |
|---|
| **Note:** Accept answers using transformation formula for product of sines if steps are shown clearly. |

| |
|---|
| **Note:** Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps. |

***[9 marks]***

*continued…*



*Question 13 continued*

(d) **EITHER**

$\sin x + \sin 3x = \cos x \Rightarrow \dfrac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1-\cos 4x = 2\sin x\cos x, \ (\sin x \neq 0)$ ***A1***

$\Rightarrow 1-\left(1-2\sin^2 2x\right) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \dfrac{1}{2}$ ***A1***

$2x = \pi,\ 2x = \dfrac{\pi}{6}$ and $2x = \dfrac{5\pi}{6}$

**OR**

$\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x\cos x = \cos x$ ***M1A1***

$\Rightarrow (2\sin 2x - 1)\cos x = 0, \ (\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \dfrac{1}{2}$ or $\cos x = 0$ ***A1***

$2x = \dfrac{\pi}{6},\ 2x = \dfrac{5\pi}{6}$ and $x = \dfrac{\pi}{2}$

**THEN**

$\therefore\ x = \dfrac{\pi}{2},\ x = \dfrac{\pi}{12}$ and $x = \dfrac{5\pi}{12}$ ***A1***

| |
|---|
| **Note:** Do not award the final ***A1*** if extra solutions are seen. |

***[6 marks]***

***Total [19 marks]***

Diploma Programme
Programme du diplôme
Programa del Diploma

**Mathematics**
**Higher level**
**Paper 3 – sets, relations and groups**

Friday 18 November 2016 (morning)

1 hour

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

4 pages

8816 – 7209




Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 13]

Let $\{G, \circ\}$ be the group of all permutations of $1, 2, 3, 4, 5, 6$ under the operation of composition of permutations.

(a) (i) Write the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 1 & 5 \end{pmatrix}$ as a composition of disjoint cycles.

(ii) State the order of $\alpha$. [3]

(b) (i) Write the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ as a composition of disjoint cycles.

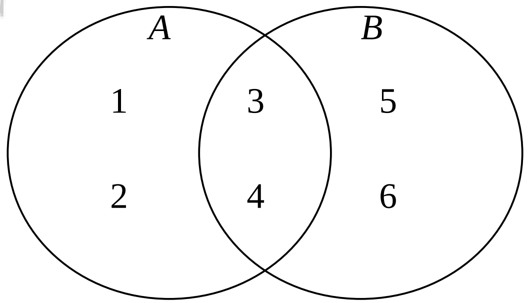
(ii) State the order of $\beta$. [2]

(c) Write the permutation $\alpha \circ \beta$ as a composition of disjoint cycles. [2]

(d) Write the permutation $\beta \circ \alpha$ as a composition of disjoint cycles. [2]

(e) State the order of $\{G, \circ\}$. [2]

Consider the following Venn diagram, where $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$.




(f) Find the number of permutations in $\{G, \circ\}$ which will result in $A$, $B$ and $A \cap B$ remaining unchanged. [2]

**2.** [Maximum mark: 21]

(a) Let $A$ be the set $\{x \mid x \in \mathbb{R}, x \neq 0\}$. Let $B$ be the set $\{x \mid x \in ]-1, +1[, x \neq 0\}$.

A function $f : A \to B$ is defined by $f(x) = \frac{2}{\pi}\arctan(x)$.

(i) Sketch the graph of $y = f(x)$ and hence justify whether or not $f$ is a bijection.

(ii) Show that $A$ is a group under the binary operation of multiplication.

(iii) Give a reason why $B$ is not a group under the binary operation of multiplication.

(iv) Find an example to show that $f(a \times b) = f(a) \times f(b)$ is not satisfied for all $a, b \in A$. [13]

(b) Let $D$ be the set $\{x \mid x \in \mathbb{R}, x > 0\}$.

A function $g : \mathbb{R} \to D$ is defined by $g(x) = e^x$.

(i) Sketch the graph of $y = g(x)$ and hence justify whether or not $g$ is a bijection.

(ii) Show that $g(a + b) = g(a) \times g(b)$ for all $a, b \in \mathbb{R}$.

(iii) Given that $\{\mathbb{R}, +\}$ and $\{D, \times\}$ are both groups, explain whether or not they are isomorphic. [8]

**3.** [Maximum mark: 15]

An Abelian group, $\{G, *\}$, has 12 different elements which are of the form $a^i * b^j$ where $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$. The elements $a$ and $b$ satisfy $a^4 = e$ and $b^3 = e$ where $e$ is the identity.

(a) State the possible orders of an element of $\{G, *\}$ and for each order give an example of an element of that order. [8]

Let $\{H, *\}$ be the proper subgroup of $\{G, *\}$ having the maximum possible order.

(b) (i) State a generator for $\{H, *\}$.

(ii) Write down the elements of $\{H, *\}$.

(iii) Write down the elements of the coset of $H$ containing $a$. [7]





**4.** [Maximum mark: 11]

A relation $S$ is defined on $\mathbb{R}$ by $aSb$ if and only if $ab > 0$.

(a) Show that $S$ is

(i) not reflexive;

(ii) symmetric;

(iii) transitive. [4]

A relation $R$ is defined on a non-empty set $A$. $R$ is symmetric and transitive but not reflexive.

(b) Explain why there exists an element $a \in A$ that is not related to itself. [1]

(c) Hence prove that there is at least one element of $A$ that is not related to any other element of $A$. [6]

Diploma Programme
Programme du diplôme
Programa del Diploma

**Mathematics**
**Higher level**
**Paper 3 – statistics and probability**

Friday 18 November 2016 (morning)

1 hour

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8816 – 7210







Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 17]

In this question you may assume that these data are a random sample from a bivariate normal distribution, with population product moment correlation coefficient $\rho$.
Richard wishes to do some research on two types of exams which are taken by a large number of students. He takes a random sample of the results of $10$ students, which are shown in the following table.

| **Student** | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** |
|---|---|---|---|---|---|---|---|---|---|---|
| **Exam 1** | 51 | 70 | 10 | 22 | 99 | 33 | 45 | 8 | 65 | 82 |
| **Exam 2** | 52 | 64 | 8 | 25 | 90 | 43 | 50 | 50 | 70 | 50 |

(a) For these data find the product moment correlation coefficient, $r$. [2]

Using these data, it is decided to test, at the $1\%$ level, the null hypothesis $H_0 : \rho = 0$ against the alternative hypothesis $H_1 : \rho > 0$.

(b) (i) State the distribution of the test statistic (including any parameters).

(ii) Find the $p$-value for the test.

(iii) State the conclusion, in the context of the question, with the word “correlation” in your answer. Justify your answer. [6]

Richard decides to take the exams himself. He scored $11$ on Exam 1 but his result on Exam 2 was lost.

(c) Using a suitable regression line, find an estimate for his score on Exam 2, giving your answer to the nearest integer. [3]

Caroline believes that the population mean mark on Exam 2 is $6$ marks higher than the population mean mark on Exam 1. Using the original data from the $10$ students, it is decided to test, at the $5\%$ level, this hypothesis against the alternative hypothesis that the mean of the differences, $d =$ exam 2 mark $-$ exam 1 mark, is less than $6$ marks.

(d) (i) State the distribution of your test statistic (including any parameters).

(ii) Find the $p$-value.

(iii) State the conclusion, justifying the answer. [6]

**2.** [Maximum mark: 17]

John rings a church bell 120 times. The time interval, $T_i$, between two successive rings is a random variable with mean of 2 seconds and variance of $\frac{1}{9}$ seconds$^2$.

Each time interval, $T_i$, is independent of the other time intervals. Let $X = \sum_{i=1}^{119} T_i$ be the total time between the first ring and the last ring.

(a) Find

(i) $\mathrm{E}(X)$;

(ii) $\mathrm{Var}(X)$. [3]

(b) Explain why a normal distribution can be used to give an approximate model for $X$. [2]

(c) Use this model to find the values of $A$ and $B$ such that $\mathrm{P}(A < X < B) = 0.9$, where $A$ and $B$ are symmetrical about the mean of $X$. [7]

The church vicar subsequently becomes suspicious that John has stopped coming to ring the bell and that he is letting his friend Ray do it. When Ray rings the bell the time interval, $T_i$, has a mean of 2 seconds and variance of $\frac{1}{25}$ seconds$^2$.

The church vicar makes the following hypotheses:

$H_0$: Ray is ringing the bell; $H_1$: John is ringing the bell.

He records four values of $X$. He decides on the following decision rule:

If $236 \le X \le 240$ for all four values of $X$ he accepts $H_0$, otherwise he accepts $H_1$.

(d) Calculate the probability that he makes a Type II error. [5]



**3.** [Maximum mark: 15]

Alun answers mathematics questions and checks his answer after doing each one. The probability that he answers any question correctly is always $\frac{6}{7}$, independently of all other questions. He will stop for coffee immediately following a second incorrect answer. Let $X$ be the number of questions Alun answers before he stops for coffee.

(a) (i) State the distribution of $X$, including its parameters.

(ii) Calculate $\mathrm{E}(X)$.

(iii) Calculate $\mathrm{P}(X = 5)$. [6]

Nic answers mathematics questions and checks his answer after doing each one. The probability that he answers any question correctly is initially $\frac{6}{7}$. After his first incorrect answer, Nic loses confidence in his own ability and from this point onwards, the probability that he answers any question correctly is now only $\frac{4}{7}$.

Both before and after his first incorrect answer, the result of each question is independent of the result of any other question. Nic will also stop for coffee immediately following a second incorrect answer. Let $Y$ be the number of questions Nic answers before he stops for coffee.

(b) (i) Calculate $\mathrm{E}(Y)$.

(ii) Calculate $\mathrm{P}(Y = 5)$. [9]

**4.** [Maximum mark: 11]

Two independent discrete random variables $X$ and $Y$ have probability generating functions $G(t)$ and $H(t)$ respectively. Let $Z = X + Y$ have probability generating function $J(t)$.

(a) Write down an expression for $J(t)$ in terms of $G(t)$ and $H(t)$. [1]

(b) By differentiating $J(t)$, prove that

(i) $\mathrm{E}(Z) = \mathrm{E}(X) + \mathrm{E}(Y)$;

(ii) $\mathrm{Var}(Z) = \mathrm{Var}(X) + \mathrm{Var}(Y)$. [10]